16 PT

Theorem. Let x be real. If $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.

Proof. By subtraction, $x^3-6x^2+9x-4=0$, which factors as $(x-1)^2(x-4)=0$. ■

14 PT

Theorem. Let x be a real number. If $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.

Proof. That $x=1$ and $x=4$ are solutions is readily verified. Suppose there is a third solution x such that $x\neq1$ and $x\neq4$. Then we may divide $x^3-6x^2+11x-6=2x-2$ by $x-1$ to obtain $x^2-5x+6=2$, that is, $x^2-5x+4=0$. Now dividing by $x-4$ leaves $x-1=0$. Dividing again by $x-1$ we conclude $1=0$, which is absurd. Hence $x=1$ or $x=4$ as required.

12 PT

Theorem. The truth value of the material implication that the algebraic equality of the univariate monic cubic quadrinomial $x^3-6x^2+11x-6$ with the univariate linear binomial $2x-2$ over the field of real numbers implies the exclusive disjunction $x=1$ or $x=4$ is True.

Proof. Suppose the existentialization of a field element x satisfying the antecedent algebraic equality. By the commutative and distributive properties of the field, the application of the polynomial subtraction operator with subtrahend $2x-2$ to the antecedent equality results in the algebraic equality of the univariate monic cubic quadrinomial x^3-6x^2+9x-4 and the additive identity element. The ensuing factorization $(x-1)^2(x-4)$ of the additive identity element necessitates, according to the nonexistence of zero divisors in a field, the consequent exclusive disjunction. QED

11 PT

Suppose x cub'd less four were equal to
Six times x squard less nine times x. Before
Deducing more, let's state up front the two
Solutions: x is one, or x is four.
Potentially, how might this manifest?
What's given can be made quadratic free,
(i.e., by change of character), depress'd:
y cub'd amounts to two and y times three.
Again, a clever transformation: add
To that the square of y on left and right.
Let not the complication make us sad;
A simple factoring puts all in sight.
Thus y is two or minus one, and more,
Back substituting, x is one or four.

10 PT

Here is a fact:
If x is real and the cube of x less six times the square of x plus five times x plus six times x less six is twice x less two, then x must be one or four.

The proof goes like this:
See, the first three terms on the left side split as the square of x less five times x all times x less one. And more, the last two terms on the left side split as six times x less one, while the right side splits as two times x less one. Thus, if x were to be one, we have nought plus nought is nought, which is true. So, x may be one. Else x is not one, and x less one is not nought. So we can times the whole thing by one on top of x less one to yield: the square of x less five times x plus six is two. Drop two from each side, and the square of x less five times x plus four is nought. Now this splits as x less four times x less one. Since we said x less one is not nought, x less four must be. So x is one or four, as was to be shown.

9 PT

Suppose the real cubic polynomial x cubed minus six x squared plus eleven x minus six equals two x minus two. Or, simply, x cubed minus six x squared plus nine x minus four equals zero. Then x has two roots, one and four. The idea of the proof is to transform the equation by a change of variable into a degree six equation that factors as a perfect square of cubes. This is usually done in two steps. First, get rid of the quadratic term by substituting y plus two in place of x. This results in the so-called depressed cubic, y cubed minus three y minus two. Second, replace y by the sum of z plus one over z. After multiplying the result by z cubed to clear the denominator, we get z to the sixth power minus two z cubed plus one. This factors nicely as z cubed minus one times z cubed minus one. So, z is a cube root of unity: either one, or minus a half plus half the square root of minus three, or minus a half minus half the square root of minus three. If z equals one, then y is two, and x, four. Taking z to be either of the two complex cube roots results in y equal to minus one, which gives the double root, x equal one. So, there are only two distinct roots of the cubic, one and four.

8 PT

Theorem (a statement derived from premises rather than assumed). The truth value (the attribute assigned as the semantic value of a proposition) of the material implication (the logical connective between two statements p and q that is equivalent to not both p and not q) that the algebraic equality (relationship between two algebraic expressions which asserts that they evaluate to the same value) of the univariate (single-variable) monic (having a leading coefficient of 1) cubic (third degree) quadrinomial (a four-term polynomial) $x^3-6x^2+11x-6$ with the univariate (single-variable) linear (first degree) binomial (two-term polynomial) $2x-2$ over the field (a nonzero commutative division ring) of real numbers (the numbers that can be expressed by a possibly infinite decimal representation) implies the exclusive disjunction (the logical operation that outputs True only when inputs differ) $x=1$ or $x=4$ is True (one of the two truth values in the Boolean domain, the other being False).

Proof (a chain of reasoning using rules of inference that leads to a desired conclusion). Suppose the existentialization (a quantification which is interpreted as “there is at least one”) of a field element (a member of a nonzero commutative division ring) x satisfying the antecedent (the first operand of a material implication) algebraic equality (relationship between two algebraic expressions which asserts that they evaluate to the same value). By the commutative (having the symmetry of a binary operation wherein the result does not depend on the order of the operands to which it is applied) and distributive (having the symmetry of an operation on a combination in which the result is the same as that obtained by performing the operation on the individual members of the combination, and then combining them) properties of the field (a nonzero commutative division ring), the application of the polynomial (an expression that is a sum of terms, each term being a product of a constant and a non-negative power of a variable or variables) subtraction operator (the inverse of the addition operator) with subtrahend (the second operand of the subtraction operator which, when added to the difference, equals the minuend) $2x-2$ to the antecedent (the first operand of a material implication) equality results in the algebraic equality (relationship between two algebraic expressions which asserts that they evaluate to the same value) of the univariate (single-variable) monic (having a leading coefficient of 1) cubic (third degree) quadrinomial (a four-term polynomial) x^3-6x^2+9x-4 and the additive identity element (the element of a set that leaves other elements unchanged when added to them, often denoted as 0). The ensuing factorization (a decomposition into a product of factors, which when multiplied together give the original) $(x-1)^2(x-4)$ of the additive identity element necessitates, according to the nonexistence of zero divisors (nonzero elements in a ring that can be multiplied with another nonzero element in the ring to form a product equal to the ring’s zero element) in a field (a nonzero commutative division ring), the consequent (the second operand of a material implication) exclusive disjunction (the logical operation that outputs True only when inputs differ). QED (quod erat demonstrandum / being what was required to prove)

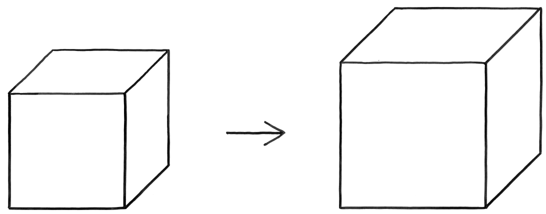
March 9, 1539

NEW SOLUTION TO ANCIENT MATHEMATICAL RIDDLE

By ANTONIO da CELLATICO

It all started with the oracle at Delphi.

In order to calm the political strife within the ancient Greek city of Delos, the oracle gave the citizens a geometry puzzle. It goes by the name of the "Delian problem," and mathematicians have been mulling it over ever since.



On December 29, an Italian mathematician from Brescia claimed to have solved a related problem that could turn out to be a breakthrough in the Delian problem.

The Brescian, Zuanne de Tonini da Coi is said to have relied on novel techniques belonging to the field known as "Algebra," though details of his solution have been slow to emerge.

Da Coi's claim appeared in a letter to the Milanese medical doctor and mathematician Hieronimo Cardano.

"It's sensational," said Cardano, a recipient of grants from the Marquis del Vasto. "If it turns out to be right, it could signal a revolution in the field."

Others are less convinced.

"Sometimes we arrive at the solution of an equation without yet being able to justify it," said Niccolò Tartaglia, an-biher mathematician from Brescia not involved in the study. "But before I see a proof, it's my habit to maintain doubt."

Near the end of the last century, Luca Pacioli, the Franciscan friar and scientific collaborator with the great Leonardo da Vinci, struggled with similar problems of Algebra. According to scientists familiar with his treatise on the matter, Pacioli even suggested that some of the so-called "cubic equations" might not have solutions.

Such doubts were shared by the Delians, too. The oracle's instruction was to measure a new altar to the god Apollo in the shape of a cube. One of Plato's favorite forms, the cube is a solid in the shape of a die with six square sides all of which are equal.

"The tricky part is that, according to the oracle's demand, the new altar must equal precisely two times the old altar," explained Dr. Cardano. "You can't simply double the side length since that would result in an altar equal eight times the first." As much happiness as an oversized altar might bring Apollo, it wouldn't satisfy the oracle.

In the elaborate terms of mathematics today, da Coi's problem appears no less baffling than that of the ancients. According to his letter, he finds "the cube and eleven times the side and two equal six times the square and twice the side and six."

And yet da Coi's equation might just have a solution.

Until this new discovery is fully resolved, research will continue. In any case, it seems likely that the Delian Problem will continue to command at least the attention of mathematicians, if not the citizens of Delos, for many years to come.

October 23, 2015, New York

The large room is dim and sparse except for the giant CT scanner at one end. The X-ray tube and detectors begin to spin within the machine's large ring as I climb onto the narrow table that extends from its gape. Having tucked me in with a blanket and foam supports behind my knees, the technologist retreats to the control room on the other side of a lead-lined wall.
If x cubed minus six x squared plus nine x minus four is zero, then x is one or four.

There's an intercom inside the doughnut hole of the scanner, and after checking in one last time the machine takes over. "Breathe in and hold," it commands from overhead, like an assertive copilot on a very small plane. I am motionless as it calibrates to my body and the database of scans-thankfully clear-since the surgery five years ago.

The product of x minus one times x minus one times x minus four is zero.

My habit is to work with pen and paper. But I've written out the solution to this equation so many times that even lying in the scanner with arms outstretched over head I can mentally retrace each step. "Breathe." As I exhale I hardly think about tumor cells.

$$x^3-6x^2+9x-4=(x-1)(x-1)(x-4)$$

My memory relies just as much if not more on the algebraic symbols themselves—their shape in my mind's eye and their names in my ear—than on the logic of the solution. But these are meager images. What I need is a geometric proof.

The table moves me into position and begins scanning my head through a thousand sections.

A long time ago, a friend suggested an "architectural" proof that shows this factorization in one or more axonometric drawings of a cube. I try to see each term of the polynomial volumetrically, as Euclid would. I try to see the triple product in the same way. The volume degenerates at the roots. I'm lost.

The scan has progressed to my torso; my head emerges from the other end of the doughnut hole. I roll my eyes to look around. Architecturally, the scanning room is more of a low box than a cube— a cube with its top layer sectioned off.

And then, without effort, the factorization appears. Let x be greater than four. To subtract six x squared, first cut four x squared off the top, then cut one x squared from the front, and finally one x squared from the side. The low box revealed within has volume $x-4$ times x-1 times $x-1$.

The problem is as good as solved. After a long beep, the machine declares: "Scan finished."

THÉORÈME.

Les racines réelles de $P(x) = (x^3-6x^2+11x-6)-(2x-2)$ sont 1 et 4.

DÉMONSTRATION.

On vérifie immédiatement que $P(1)=P(4)=0$. Comme le polynôme $P \in R[x] \subset C[x]$ est de degré 3, d'après le théorème de d'Alembert on sait qu'il admet au maximum 3 racines réelles. Nous raisonnons par l'absurde en supposant que $a \in R$ soit une troisième racine distincte. Alors,

$$P(x)=(x-1)(x-4)(x-a)=x^3-(5+a)x^2+(4+5a)x-4a.$$

On déduit de la seconde égalité que $a=1$, ce qui nous donne une contradiction. Donc, les racines de P sont 1 et 4. CQFD

SATZ.

Sei x ein ganze Zahl. Wenn $x^3-6x^2+11x-6=2x-2$ ist, dann ist $x=1$ oder $x=4$.

BEWEIS.

Sei p das Polynom dritten Grades gegeben durch $p(x)=(x^3-6-{}^2+11x-6)-(2x-2) \in Z[x]$. Wir wenden die Kroneckersche Methode an, um ein Polynom aus G[x] in Primfaktoren zu zerlegen, wo G ein Gauß'scher Ring ist. Soll nun $p(x)$ durch die Linearform $q(x)=x-a$ teilbar sein, so muss $p(x_0)$ durch $q(x_0)$ und $p(x_1)$ durch $q(x_1)$ teilbar sein. Jedes $p(x_i)$ in Z besitzt aber nur endlich viele Teiler. Setzen wir $x_0=2$, so ist $q(x_0)=2-a$ ein Faktor von $p(2)=-2$. Da ± 1 und ± 2 die einzigen Faktoren von -2 sind, folgt, dass a ein Element der Menge $M_0=\{0,1,3,4)$ sein muss. Wenn allerdings $x_1=3$ ist, dann ist 3-a einer der Faktoren ± 1, ± 2 oder ± 4 von $p(3)=-4$, m.a.W., a liegt in $M_1=(-1,1,2,4,5,7\}$. Da eine Wurzel a von $p(x)$ beide Bedingungen erfüllen muss, schließen wir, dass $a \in M_0 \cap M_1=\{1,4\}$, wie behauptet.

p^3/q^2	x_1	x_2	x_3
-6.95	0.8871209	1.0867409	4.0261382
-6.94	0.8903176	1.0848419	4.0248406
-6.93	0.8935810	1.0828769	4.0235420
-6.92	0.8969170	1.0808405	4.0222425
-6.91	0.9003320	1.0787260	4.0209420
-6.90	0.9038335	1.0765259	4.0196406
-6.89	0.9074304	1.0742315	4.0183381
-6.88	0.9111331	1.0718322	4.0170347
-6.87	0.9149541	1.0693157	4.0157303
-6.86	0.9189084	1.0666667	4.0144249
-6.85	0.9230149	1.0638666	4.0131185
-6.84	0.9272970	1.0608918	4.0118112
-6.83	0.9317850	1.0577122	4.0105028
-6.82	0.9365190	1.0542876	4.0091935
-6.81	0.9415540	1.0505629	4.0078831
-6.80	0.9469694	1.0464588	4.0065718
-6.79	0.9528870	1.0418536	4.0052594
-6.78	0.9595117	1.0365423	4.0039461
-6.77	0.9672434	1.0301249	4.0026317
-6.76	0.9771147	1.0215689	4.0013164
-6.75	1.0000000	1.0000000	4.0000000
-6.74	1.0006587 - 0.0222173i	1.0006587 + 0.0222173i	3.9986826
-6.73	1.0013179 - 0.0314132i	1.0013179 + 0.0314132i	3.9973642
-6.72	1.0019776 - 0.0384646i	1.0019776 + 0.0384646i	3.9960448
-6.71	1.0026378 - 0.0444053i	1.0026378 + 0.0444053i	3.9947244
-6.70	1.0032985 - 0.0496356i	1.0032985 + 0.0496356i	3.9934029
-6.69	1.0039598 - 0.0543611i	1.0039598 + 0.0543611i	3.9920805
-6.68	1.0046215 - 0.0587036i	1.0046215 + 0.0587036i	3.9907570
-6.67	1.0052838 - 0.0627428i	1.0052837 + 0.0627428i	3.9894325
-6.66	1.0059466 - 0.0665340i	1.0059466 + 0.0665340i	3.9881069
-6.65	1.0066098 - 0.0701173i	1.0066098 + 0.0701173i	3.9867803
-6.64	1.0072737 - 0.0735232i	1.0072737 + 0.0735232i	3.9854527
-6.63	1.0079380 - 0.0767753i	1.0079380 + 0.0767753i	3.9841240
-6.62	1.0086028 - 0.0798923i	1.0086028 + 0.0798923i	3.9827943
-6.61	1.0092682 - 0.0828895i	1.0092682 + 0.0828895	3.9814636
-6.60	1.0099341 - 0.0857794i	1.0099341 + 0.0857794i	3.9801318
-6.59	1.0106005 - 0.0885726i	1.0106005 + 0.0885726i	3.9787990
-6.58	1.0112675 - 0.0912778i	1.0112675 + 0.0912778i	3.9774651
-6.57	1.0119350 - 0.0939028	1.0119350 + 0.0939028i	3.9761301
-6.56	1.0126029 - 0.0964540i	1.0126029 + 0.0964540i	3.9747941